International Journal of Theoretical Physics, Vol. 36, No. 11, 1997

Interior Symmetries of Hadrons: $SO(3,2)$ as a Spectrum-Generating Group

Sujeewa Wickramasekara[1]

Received June 25, 1997

In this work, we study some applications of the spectrum-generating group (SGG) formalism to obtain the mass-spin spectra of hadrons. The possibility of classifying (medium-energy) hadrons in terms of a symmetry group defining the center-of-mass motion and an SGG defining the interior motion is discussed. After considering the defining commutation relations and equations of motion of the generators of the SGG, it is shown how the hadronic spectral information is obtained through a Hamiltonian that is a constraint relation between the generators of the symmetry group and those of the SGG.

1. INTRODUCTION

It is the received point of view that a microphysical object is elementary in a particular energy range of interest if it carries no manifest internal structure in that range. The motions that such an object may perform are understood in quantum mechanics as bringing forth the group of symmetry transformations. Recall that a symmetry transformation is that which links observations of different, equivalent observers who look at the same physical system. In the linear topological vector space $\mathcal{H}$, i.e., a Hilbert space or a dense subspace thereof, whose elements represent the states of the physical system, the symmetry transformations $R(\alpha)$ are represented by the linear operators $U(R(\alpha))$. In this notation, α denotes a set of continuous parameters satisfying the condition $R(O) = I$, the identity operator. Thereupon, the generators O_i of the representation $U(R(\alpha))$ of G, defined by

$$O_i = i\,\frac{\partial U(R(\alpha))}{\partial \alpha_i}\bigg|_{\alpha=0}$$

[1] Department of Physics, University of Texas, Austin, Texas 78712.

0020-7748/97/1100-2409$12.50/0

acquire physical content as the observables pertaining to the microphysical object. As a consequence of the property that the physical states are represented not by the vectors but rather by the rays of the space $\mathcal{H}$, the mapping $R(\alpha) \rightarrow U(R(\alpha))$ furnishes a linear, ergo unitary (Wigner, 1939), *projective* representation of the symmetry group G. According to Wigner's pioneering work, relativistic elementary particles, in particular, are classified according to the continuous unitary irreducible projective representations of the Poincaré group $\mathcal{P}$. The eigenvalues m^2 and $j\ (j + 1)$ of, respectively, the invariant operators M^2 and $\hat{W}$ of $\mathcal{P}$ uniquely characterize these representation spaces $\mathcal{H}(m, j)$ of the group on any future-directed orbit in the Minkowski space. Along with the physical interpretation of m and j as the mass and the spin of the relativistic elementary quantum object, the two parameters which characterize such objects, the generators P_μ and $J_{\mu\nu}$ of the Poincaré group are accorded meaning as the momentum and angular momentum operators.

However, most microphysical systems, such as molecules, nuclei, and hadrons, manifest the behavior of extended quantum physical objects when the energy is sufficiently high. In the view adopted here, hadrons, in particular, are considered as extended relativistic objects. In this limit the symmetry group G of the relevant space-time acquires reality as the group formed by the motions of center of mass of the extended object. In addition, the extended object also performs interior collective motions and the motions of the constituents: the diatomic molecule can execute rotations about its center of mass, vibrations along its internuclear axis, while its electrons can rapidly rotate or spin about the direction of the internuclear axis (the rotating-vibrating dumbbell with a flywheel). The group that describes these interior motions, in contrast to the center-of-mass motions, is the Spectrum-Generating Group (SGG), which is also known, perhaps more commonly, as the Dynamical Group.

The initial physical motivation for the SGG (in the 1960s) for the mass spectrum was the enormous proliferation of hadrons. Hadrons exist with rather high angular momentum, and some masses follow a simple empirical formula $m^2 = m_0^2 + (1/\alpha')j$, where (m,j) = (mass, spin), and $\sqrt{\alpha'}$ is a constant with dimension of length—an *elementary length*. If the mass measured in units of GeV is given by the spin measured in units of 1 (for $\hbar$ = 1, c = 1), or by any other dimensionless (for $\hbar = c = 1$) observable, then there must exist a constant of dimension cm that converts the units. This is analogous to c being the constant of nature that converts time measured in units of cm into time measured in seconds $x = ct$, and to h being the constant that converts energy units into frequency units $E = h\nu$. The idea that, in addition to the constants c and $\hbar$, there must exist a third fundamental constant, an elementary length ℓ, in fact, goes back to Heisenberg (1936). For the hadron spectrum the elementary length must be $\ell \approx 10^{-13}$ cm = 1 fm; $\sqrt{\alpha'} = 0.19$ fm or $1/\sqrt{\alpha'} \approx 1$ GeV.

The large number of hadrons discovered and the relations observed among them such as the above mass-spin formula do not permit one to view them all as truly elementary particles. Instead, one is led to the possibility that they are various states of one single structured relativistic quantum system. Thus, the hadron (mass) spectrum can be connected to a hadron structure. This is a rather familiar idea in molecular and nuclear physics. The structure of molecules and nuclei is understood in two complementary ways: in terms of their constituents and in terms of their (collective) motions. In low-energy molecules and nuclei, the dominant feature is the collective motions, while the constituent excitations become important only at much higher energies. Similarly, hadrons (just like molecules in the $<10^{-4}$ eV region) behave like elementary (pointlike) particles at very low energies, and like composite aggregates of elementary pointlike constituents (partons, quarks) at very high energies. The standard model applicable in the high-energy domain (or large-momentum-transfer limit) is QCD, which explains the *unstring-like* behavior of hadrons at short distances in terms of a small number of pointlike constituents. At intermedeate energies, however, hadrons can be seen to demonstrate the properties expected of extended (relativistic) objects performing collective motions. The motions of the center of mass of an extended object, just like those of an elementary particle, form the symmetry group, the generators of which now describe the center-of-mass observables. The interior, collective motions (rotations and vibrations) form the spectrum-generating group, and their generators give the interior observables. Thus, by a spectrum-generating group we mean a group (in general a noncompact one) which gives the energy or mass spectrum of a quantum mechanical system (Barut and Bohm, 1965).

The spectrum-generating group, in this light, displays the interior symmetries arising from the interior collective motions of the extended object. The properties of the generators of the SGG, which then represent the interior observables such as distances (electric dipole operators), quadrupole operators, and interior (relative) momenta, are not obvious and sometimes quite surprising. For instance, the components of the intrinsic momenta (also the distances) need not commute among themselves because the fast motion of the constituents (partons, quarks, dyons) can induce into the dynamics of the slow collective motion gauge potentials which lead to noncommuting covariant momenta, etc. (Bohm *et al.*, 1992). As a consequence, the choice of the group for interior motions, the SGG, is also not obvious and depends upon the properties of each particular extended object.

The interior observables cause transitions between different vibrational and rotational states of the extended object (e.g., dipole operators describe transitions between energy eigenstates). These different (vibrational and rotational) energy levels of the intrinsic degrees of freedom are labeled by discrete quantum numbers such as ν (for vibrational excitations) and j (for rotational excitations). In the case of hadrons these excited states (ν, j) are the different hadrons described by different irreducible representation spaces of the symmetry group $\mathcal{P}$, and therefore these spaces $\mathcal{H}(m, j)$ must be also labeled by these quantum numbers: $\mathcal{H}_\nu^{(m,j)} = \mathcal{H}(m(\nu, j), j)$. The level splitting (mass spectrum) $m = m(\nu, j)$ between the levels labeled by ν (vibrational quantum number) and j (rotational quantum number) is given by the elementary length $\ell = \sqrt{\alpha'} \approx 0.2 \times 10^{-13}$ cm through, e.g., formulas like $m^2 = m_0^2 + (1/\alpha')\nu$. For the states with $\nu = j$ (yrast states in the terminology of nuclear physics) one is back to the empirical formula for the linear Regge trajectory mentioned above. This identifies the hadrons on a Regge trajectory with the *yrast states* of a relativistic vibrating rotator (see Section 4).

The irreducible representation spaces $\mathcal{H}^\nu(m(\nu, j), j)$ of the group describing the motions of the center of mass of an extended object are associated with different states (subspaces labeled by (ν, j)) of an irreducible representation space of the SGG. Hence, a theory of extended relativistic quantum objects can be arrived at through a certain careful union of the symmetry group $G = \mathcal{P}$ and the spectrum-generating group.

2. INTERIOR OBSERVABLES AND SPECTRUM-GENERATING GROUP

As mentioned above, the generators of the Lie algebra of SGG represent the observables which describe the interior motion of the extended relativistic quantum system. The choice of these observables and the corresponding SGG depends on the way one specifies and treats both the interior motions themselves and their essential nonseparability in the relativistic limit from the motions of the center of mass. In this section we postulate the interior observables, and in the next we discuss the Lie algebra of the SGG they generate.

It is recalled that in the theory of the nonrelativistic quantum rotator (Bohm, 1979) an operator $\vec{D}$, a vector with respect to $SO(3)_{S_{ij}}$, is introduced to represent the symmetry axis. Here, S_{ij} are the generators of the Lie algebra of $SO(3)$, the group of rotations about the center of mass (c.m.), and they satisfy the commutation relations (c.r.)

$$[S_{ij}, S_{kl}] = -i(g_{ik}S_{jl} + g_{jl}S_{ik} - g_{il}S_{jk} - g_{jk}S_{il}) \tag{2.1}$$

The interior position operators D_i transform between different irreducible representations R^s of $SO(3)$. Since $\vec{D}$ is taken to be a vector operator, the D_i fulfill with the S_{ij} the following c.r.:

$$[D_i, S_{jk}] = -i(g_{ij}D_k - g_{ik}D_j) \tag{2.2}$$

If the rotator consists of charges $+e$ and $-e$ centered respectively at $\vec{Q}_{(+)}$ and $\vec{Q}_{(-)}$, then $\vec{D}$ is simply the vector $(\vec{Q}_{(+)} - \vec{Q}_{(-)})$ and $e\vec{D}$ gives the dipole moment. To determine the structure of the group that D_i and S_{ij} generate, we have to determine the c.r. for the D_i. Conventionally, the interior position operators D_i and their conjugate momenta $\Pi_i = (m_{(-)}P_{(+)i} - m_{(+)}P_{(-)i})/(m_{(+)} + m_{(+)})$ are taken to satisfy the canonical commutation relations

$$[D_i, D_j] = 0, \qquad [\Pi_i, \Pi_j] = 0, \qquad [\Pi_i, D_j] = i\hbar\delta_{ij} \tag{2.3a}$$

On the basis of (2.1), (2.2), and (2.3a), it is generally concluded that $E(3)$, whose generators are D_i and S_{ij}, is the SGG for the nonrelativistic rotator. The commutation relations (2.3a) can be derived from the assumption that the operators $P_{(\pm)i}$ and $Q_{(\pm)i}$ satisfy the following commutation relations:

$$[P_{(\pm)i}, Q_{(\pm)j}] = i\hbar\delta_{ij}, \qquad [P_{(\pm)i}, Q_{(\mp)j}] = [P_{(+)i}, P_{(-)j}] = [Q_{(+)i}, Q_{(-)j}] = 0$$

This would mean that the centers of positive and negative charge behave like unconstrained mass points, independently governed by two Galilei groups $\mathscr{G}$. This is unlikely to be fulfilled for points inside an extended object, and it is therefore necessary to consider other candidates for SGG in addition to $E(3)$. This also means that, for the interior operators the conventional c.r. (2.3a) would have to be abandoned. Two alternatives that have been considered are the following:

$$[D_i, D_j] = -iS_{ij} \tag{2.3b}$$

$$[D_i, D_j] = iS_{ij} \tag{2.3c}$$

Commutators (2.3b) and (2.3c), together with (2.1) and (2.2), respectively lead to the groups $SO(3, 1)_{D_i,S_{ij}} \supset SO(3)_{S_{ij}}$ and $SO(4)_{D_i,S_{ij}} \supset SO(3)_{S_{ij}}$ as the corresponding spectrum-generating group. The use of $SO(3, 1)_{D_i,S_{ij}}$ as the SGG was suggested for the first time in Barut and Bohm (1965) and we shall encounter it again as the relativistic SGG for hadrons in the c.m. rest frame [see (4.48a)]. The other choice, $SO(4)_{D_i,S_{ij}}$, has also been used in nuclear physics (Iachello *et al.*, 1982).

The Hamiltonian for the nonrelativistic rotator is given by

$$H = \frac{\vec{P}^2}{2M} + \frac{\vec{S}^2}{2I} \tag{2.4a}$$

where P_i are the generators of the symmetry group $\mathscr{G}$ describing the c.m. motion. The expression (2.4a) for the energy in terms of the generators of the center of mass symmetry group and the spectrum-generating group leads to the energy spectrum of the rotating diatomic molecule:

$$E = \frac{1}{2I} s(s + 1) + \text{kinetic energy of c.m.} \tag{2.4b}$$

As mentioned in Section 1, the interior operators cause transitions between different physical states of the extended object. Operators D_i indeed perform such transitions between different s-states, and the transition probabilities are proportional to the matrix elements $|\langle s+1, s_3|\vec{D}|ss_3'\rangle|^2$.

In this light, we expect to obtain the hadron spectrum by finding an SGG. We seek for the interior position (momentum) operator a relativistic generalization of the operator $\vec{D}(\vec{\Pi})$. As seen above in (2.3) and as mentioned in Section 1, the choice is not obvious even in the nonrelativistic case, since, due to the fast motion of the constituents and the resulting induced gauge potentials, the gauge-covariant momenta may not commute (Bohm *et al.*, 1992). Therefore in the relativistic case also we should not expect that the components of the interior positions and momenta commute. We favor for the interior position a definition that has its origin in the *Zitterbewegung* of a classical relativistic object (Corben, 1968; Mathison, 1937). This interior position is given by

$$d_\mu = -S_{\mu\nu}\hat{P}^\nu M^{-1} \qquad (\mu, \nu = 0, 1, 2, 3) \tag{2.5}$$

where $\hat{P}^\nu = P^\nu M^{-1}$. The operators P^ν and $M = (P_\mu P^\mu)^{1/2}$ are respectively the c.m. momentum operator and the mass operator. This definition of the operator M requires that the P_μ fulfill the condition $P_\mu P^\mu > 0$ (see Section 4). $S_{\mu\nu}$ are the relativistic generalizations of operators S_{ij} defined in (2.1), and can be interpreted as the generators of the Lie algebra of an interior proper Lorentz group $SO(3,1)_{S_{\mu\nu}}$ if we require that they satisfy the commutation relations

$$[S_{\mu\nu}, S_{\rho\sigma}] = -i(g_{\mu\rho}S_{\nu\sigma} + g_{\nu\sigma}S_{\mu\rho} - g_{\mu\sigma}S_{\nu\rho} - g_{\nu\rho}S_{\mu\sigma}) \tag{2.6}$$

In analogy to the nonrelativistic relations $J_{ij} = L_{ij} + S_{ij}$ and the 4-dimensional Dirac case $J_{\mu\nu} = L_{\mu\nu} + \frac{1}{2}\sigma_{\mu\nu}$, we also require the infinite-dimensional operators $S_{\mu\nu}$ to appear in the generators $J_{\mu\nu}$ [which themselves satisfy the c.r. (2.6)] of the *physical* Lorentz group as

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \tag{2.7a}$$

This is motivated by the expression for the classical quantities

$$J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu + S_{\mu\nu} \tag{2.7b}$$

As in the 4-dimensional case, we may require that (2.7a) hold only in the spinor basis. Furthermore, we require that the operators $L_{\mu\nu}$ and $S_{\mu\nu}$ in (2.7a) fulfill the following conditions:

$$[L_{\mu\nu}, S_{\rho\sigma}] = 0 \qquad \text{and} \qquad \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^{\mu}L^{\rho\sigma} = 0 \tag{2.7c}$$

An immediate consequence of (2.6) and definition (2.5) of d_μ is the commutation relation

$$[d_\mu, d_\nu] = \frac{-i}{M^2}(S_{\mu\nu} + d_\mu P_\nu - d_\nu P_\mu) \tag{2.8}$$

Thus our interior position operators d_μ—for that matter, even their components on a 3-dimensional spacelike hypersurface—do not commute. However, in the nonrelativistic contraction limit $c \to \infty$, the d_m go over into commuting operators d_m^∞ (and $d_0^{(\infty)} = 0$) (Aldinger *et al.*, 1984). This limiting case, in fact, is one justification of the interpretation of d_μ as the interior position operators.

The position X_μ that appears in (2.7b) can now be given in terms of d_μ and the mean position of the extended object Y_μ:

$$X_\mu(\tau) = Y_\mu(\tau) + d_\mu \tag{2.9}$$

where $\dot{Y}_\mu = \hat{P}_\mu = P_\mu M^{-1}$.

A straightforward calculation shows that the expression on the right in (2.8) is related to the spin tensor $\Sigma_{\mu\nu}$ of the Poincaré group:

$$[d_\mu, d_\nu] = -\frac{i}{M^2}\Sigma_{\mu\nu} \tag{2.10}$$

where $\Sigma_{\mu\nu} = \check{g}^{\rho}_{\mu}\check{g}^{\sigma}_{\nu}S_{\rho\sigma}$; $\check{g}_{\mu\nu} = \eta_{\mu\nu} - \hat{P}_\mu\hat{P}_\nu$, and diag. $\eta_{\mu\nu} = (+1, -1, -1, -1)$.

If the spin is not to be restricted to just one value, but allowed to have a nontrivial spectrum as on a Regge trajectory, then an operator is needed to describe the observable that performs transitions between different spin states, i.e., between two or more irreducible representations spaces $\mathcal{H}(m, j)$ of the symmetry group $\mathcal{P}$. Such an operator cannot clearly be constructed in terms of the generators of the algebra of $\mathcal{P}$. Thus, in analogy to the Dirac γ_μ matrices, which serve a similar purpose for the theory of the electron, we choose for this observable a Hermitian vector operator Λ_μ which, together with $S_{\mu\nu}$ (like γ_μ and $\sigma_{\mu\nu}$ of Dirac) forms the Lie algebra of the group $SO(3, 2)_{\Lambda_\rho, S_{\mu\nu}}$. This infinite-dimensional generalization of the Dirac case is our choice for the relativistic SGG. Along with (2.6), we then have the following defining commutation relations for the Lie algebra of $SO(3, 2)$, the SGG of our relativistic extended quantum system:

$$[S_{\mu\nu}, \Gamma_\rho] = i(g_{\nu\rho}\Gamma_\mu - g_{\mu\rho}\Gamma_\nu) \tag{2.11}$$

$$[\Gamma_\mu, \Gamma_\nu] = -iS_{\mu\nu} \tag{2.12}$$

By (2.11), Γ_μ is a Lorentz vector operator with respect to the group $SO(3, 1)_{S_{\mu\nu}}$.

The interior (noncommuting) momenta can now be defined in terms of Γ_μ by

$$\pi_\mu = \frac{-1}{\alpha'}\frac{1}{M}\breve{g}^\sigma_\mu\Gamma_\sigma \tag{2.13}$$

where α' is the constant discussed in Section 1. For a justification of this definition we have to consider the relativistic Hamiltonian, the subject of Section 4.

3. MATHEMATICS OF $SO(3, 2)$

3.1. Lie Algebra of $SO(3, 2)$

In place of the operators $S_{\mu\nu}$ and Γ_μ of Section 2 satisfying the commutation relations (2.6), (2.11), and (2.12), we may consider the operators $S_{AB} = -S_{BA}$ fulfilling the commutation relations

$$[S_{AB}, S_{CD}] = -i(g_{AC}S_{BD} + g_{BD}S_{AC} - g_{AD}S_{BC} - g_{BC}S_{AD}) \tag{3.1}$$

where $(A, B = 0, 1, 2, 3, 5)$ and diag.$g_{AB} = (+1, -1, -1, -1, +1)$.

The operators S_{AB} generate the $SO(3, 2)$ group and are called the Hermitian basis for the Lie algebra of $SO(3, 2)$.

The $SO(3, 2)$ has two Casimir operators constructed from these generators: the quadratic Casimir operator

$$C_2 \equiv \frac{1}{2}S_{AB}S^{AB} \tag{3.2}$$

and the fourth-order Casimir operator

$$C_4 \equiv -W_AW^A \tag{3.3}$$

where $W^A = \frac{1}{8}\epsilon^{ABCDE}S_{BC}S_{DE}$. Identifying $S_{\mu5} \equiv \Gamma_\mu$, where $(\mu = 0, 1, 2, 3)$, we can immediately obtain from (3.1) the commutation relations (2.6), (2.11), and (2.12). We may also define

$$S_{0i} \equiv K_i \qquad \text{and} \qquad S_i \equiv \frac{1}{2}\epsilon_{ijk}S_{jk} \tag{3.4}$$

and thereupon from (3.1) we obtain the following commutation relations:

$$[S_i, S_j] = i\epsilon_{ijk}S_k \tag{3.5a}$$

$$[S_i, K_j] = i\epsilon_{ijk}K_k \tag{3.5b}$$

$$[K_i, K_j] = -i\epsilon_{ijk}S_k \tag{3.5c}$$

$$[S_i, \Gamma_j] = i\epsilon_{ijk}\Gamma_k \tag{3.5d}$$

$$[\Gamma_i, \Gamma_j] = -i\epsilon_{ijk}S_k \tag{3.5e}$$

$$[K_i, \Gamma_j] = -i\delta_{ij}\Gamma_0 \tag{3.5f}$$

$$[S_i, \Gamma_0] = 0 \tag{3.5g}$$

$$[K_i, \Gamma_0] = -\Gamma_i \tag{3.5h}$$

$$[\Gamma_i, \Gamma_0] = K_i \tag{3.5i}$$

From (3.3a) we see that the operators S_i generate an $SO(3)$ group. By (3.5b) and (3.5d), the K_i and Γ_i are vector operators with respect to this group, and by (3.5g), Γ_0 is a scalar operator. It is also clear from these commutation relations that both the K_i and the Γ_i, together with the S_i, separately generate two $SO(3, 1)$ groups, which can be denoted respectively by $SO(3, 1)$ and $SO(3, 1)_{S_{ij},\Gamma_i}$.

Although for an nth-rank compact group the eigenvalues of the n invariant operators uniquely specify the irreducible representations (irreps) of the group, for a noncompact group, in general, the invariant operators are not sufficient to distinguish the irreps. For $SO(3, 2)$, in particular, a third label, the minimum (or maximum) eigenvalue of Γ_0, depending on whether Γ_0 is bounded from below (or above), is generally needed.

For the purposes of this paper, we are interested only in the singleton, i.e., the multiplicity-free, irreps of $SO(3, 2)$.

3.2. Reduction Chains of $SO(3, 2)$

There are three subgroup chains which are convenient for discussing the irreps of the $SO(3, 2)_{S_{AB}}$:

$$SO(2)_{S_{12}} \subset SO(3)_{S_{ij}} \subset SO(3, 1)_{S_{\mu\nu}} \subset SO(3, 2)_{S_{AB}} \tag{3.6}$$

$$SO(2)_{S_{12}} \subset SO(3)_{S_{ij}} \subset SO(3, 1)_{S_{ij},\Gamma_i} \subset SO(3, 2)_{S_{AB}} \tag{3.7}$$

$$SO(2)_{S_{12}} \subset SO(3)_{S_{ij}} \subset SO(3)_{S_{ij}} \otimes SO(2)_{\Gamma_0} \subset SO(3, 2)_{S_{AB}} \tag{3.8}$$

The reduction chains (3.6) and (3.7) are mathematically equivalent, but physically different. In the following subsection we will first discuss in some detail a particular class of representations based on the reduction chain (3.6), and then briefly outline how some singleton irreducible representations based on the reduction chain (3.8) are obtained.

3.3. Irreducible Majorana Representations of $SO(3, 2)$

Consider the reduction chain (3.6). Since we are interested only in the singleton irreps, a canonical set of commuting operators (c.s.c.o) can be chosen to consist of the invariant operators of the subgroups in the reduction chain. Hence a c.s.c.o. would be

$$C_1 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}, \qquad C_2 = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_{\mu\nu} S_{\rho\sigma}, \qquad \vec{S}^2, \qquad S_{12}, \qquad \Gamma_0 = S_{05} \tag{3.9}$$

where C_1 and C_2 are the two Casimir operators of $SO(3, 1)_{S_{\mu\nu}}$. Without further specifications, there are still many representations of $SO(3, 2)$, and additional restrictions are necessary, as we consider only singleton representations. One class of infinite-dimensional irrep that is of interest (see relativistic quantum rotator, Section 4) can be obtained by means of the constraint relation

$$\{\Gamma_\mu, \Gamma^\rho\} + \{S_{\mu\nu}, S^{\rho\nu}\} = -\delta_\mu^\rho \tag{3.10}$$

or equivalently,

$$\{S_{AB}, S^{CB}\} = -\delta_A^C \tag{3.11}$$

The representations which fulfill this condition are called irreducible Majorana representations or, following Dirac, *remarkable* representations. They are isomorphic to the representations of $SO(3, 2)$ for which the generators are realized by a pair of boson operators. The main feature of these representations is that each contains only one irrep of the $SO(3, 1)_{S_{\mu\nu}}$ subgroup.

It is recalled that the linear irreducible representations of $SO(3, 1)_{S_{\mu\nu}}$ are characterized by two numbers (k_0, c), where k_0 is an integer or half-odd integer and c an arbitrary complex number. In terms of k_0 and c, the Casimir operators are given by

$$C_1 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = k_0^2 + c^2 - 1 \tag{3.12}$$

$$C_2 = \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_{\mu\nu} S_{\rho\sigma} = ik_0 c \tag{3.13}$$

Setting $\rho = \mu$ in (3.10) yields

$$\{\Gamma_\mu, \Gamma^\mu\} + \{S_{\mu\nu}, S^{\mu\nu}\} = -4 \qquad \text{or} \qquad \Gamma_\mu \Gamma^\mu + S_{\mu\nu}, S^{\mu\nu} = -2 \tag{3.14}$$

Setting $A = C$ in (3.11) yields

$$\{S_{AB}, S^{AB}\} = -5 \qquad \text{or} \qquad \Gamma_\mu \Gamma^\mu + \frac{1}{2} S_{\mu\nu} S^{\mu\nu} = -\frac{5}{4} \tag{3.15}$$

From these it follows that

$$C_1 \equiv S_{\mu\nu} S^{\mu\nu} = S_{oi} S^{oi} + \frac{1}{2} S_{ij} S^{ij} = -\frac{3}{4} \tag{3.16}$$

$$\Gamma_\mu \Gamma^\mu = -\frac{1}{2} \tag{3.17}$$

and

$$C_2 \equiv \frac{1}{8} \epsilon^{\mu\nu\rho\sigma} S_{\mu\nu} S_{\rho\sigma} = 0 \tag{3.18}$$

These results can be used to solve (3.12) and (3.13) for k_0 and c:

$$(k_0, c) = \left(0, \frac{1}{2}\right) \tag{3.19}$$

or

$$(k_0, c) = \left(\frac{1}{2}, 0\right) \tag{3.20}$$

(3.19) implies a reduction of the representation space of $SO(3, 1)$ in terms of the integer-j irreducible representation spaces R^j of the rotation group, while (3.2a) implies a reduction in terms of the half-odd-integer irrep spaces R^j. Since the generators of $SO(3, 2)$ contain no operator which transforms between the states of integer and half-odd-integer angular momentum states, these results lead to the conclusion that the irrep space $\mathcal{H}^{SO(3,2)}$ of the group $SO(3, 2)$ reduces either as

$$\mathcal{H}^{SO(3,2)} \overset{SO(3,1)_{S_{\mu\nu}}}{=} \mathcal{H}\left(k_0 = 0, c = \frac{1}{2}\right) \overset{SO(3)_{S_{ij}}}{=} \sum_{j=k_0,k_0+1,\ldots} \oplus \mathcal{R}^j \tag{3.21}$$

or as

$$\mathcal{H}^{SO(3,2)} \overset{SO(3,1)_{S_{\mu\nu}}}{=} \mathcal{H}\left(k_0 = \frac{1}{2}, c = 0\right) \overset{SO(3)_{S_{ij}}}{=} \sum_{j=k_0,k_0+1,\ldots} \oplus \mathcal{R}^j \tag{3.22}$$

The symbol $\overset{G}{=}$ means that the spaces are equal when the transformations are restricted to the subgroup G. From these reductions we see that the basis vectors can be characterized by j and j_3, where $j = 0, 1, 2, 3, \ldots$ for (3.21) and $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ for (3.22). We denote them by $|{}^{j}_{j_3})$, and in this basis the c.s.c.o. chosen above has the following eigenvalues:

$$S_3\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = j_3\left|\begin{matrix} j \\ j_3 \end{matrix}\right)$$

$$\vec{S}^2\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = j(j+1)\left|\begin{matrix} j \\ j_3 \end{matrix}\right)$$

$$C_1\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = (k_0^2 + c^2 - 1)\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = -\frac{3}{4}\left|\begin{matrix} j \\ j_3 \end{matrix}\right)$$

$$C_2\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = ik_0c\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = 0$$

Another consequence of the condition (3.10) is that the matrix elements of Γ_0 are determined up to a sign by the matrix elements of $\vec{S}^2$. This can be easily seen if we set $\rho = \mu = 0$ in (3.10) and solve for Γ_0 the resulting equation

$$2\Gamma_0^2 + 2S_{0i}S^{0i} = -1 \tag{3.23}$$

with the equation (3.16). Thus, $\Gamma_0^2 = \vec{S}^2 + \frac{1}{4}$ and

$$\Gamma_0^2\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = \left(\vec{S}^2 + \frac{1}{4}\right)\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = \left(j(j+1) + \frac{1}{4}\right)\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = \left(j + \frac{1}{2}\right)^2\left|\begin{matrix} j \\ j_3 \end{matrix}\right) \tag{3.24}$$

The eigenvalues of Γ_0, therefore, are $\pm(j + \frac{1}{2})$. Therewith, we have obtained from the reduction chain (3.6) four infinite-dimensional Majorana irreducible representations of the group $SO(3, 2)_{S_{\mu\nu},\Gamma_\mu}$ characterized by the sign (positive or negative) of the eigenvalues of Γ_0 combined with the integer or half-odd-integer values of j. In these representations the vectors $|{}^{j}_{j_3})$ have the following transformation property under an element Λ of the Lorentz group $SO(3, 1)_{S_{\mu\nu}}$:

$$U(\Lambda)\left|\begin{matrix} j \\ j_3 \end{matrix}\right) = \sum_{\substack{-j' \le j_3 \le j' \\ j' = k_0, k_0+1, \ldots}} \left|\begin{matrix} j' \\ j_3' \end{matrix}\right) d^{(k_0,c)}{}^{j'j}_{j_3'j_3}(\Lambda) \tag{3.25}$$

where

$$U(\Lambda(\omega)) = e^{i\omega^{\mu\nu}S_{\mu\nu}} \quad \text{and} \quad \mathrm{D}^{(k_0,c)}{}^{j'j}_{j_3'j_3}(\Lambda) = \left(\begin{matrix} j' \\ j_3' \end{matrix}\right|U(\Lambda)\left|\begin{matrix} j \\ j_3 \end{matrix}\right)$$

3.4. A Summary of Singleton Representations of $SO(3, 2)$ Which Contain the Maximally Compact Subgroup $SO(3) \otimes SO(2)_{\Gamma_0}$

Next we will consider the reduction chain (3.8) that uses the maximally compact subgroup $SO(3)_{S_{ij}} \otimes SO(2)_{\Gamma_0}$ of the group $SO(3, 2)_{S_{\mu\nu},\Gamma_\mu}$. Again, we are interested only in certain singleton irreducible representations of $SO(3, 2)_{S_{\mu\nu},\Gamma_\mu}$, i.e., irreps in which an irrep of $SO(3)_{S_{ij}} \otimes SO(2)_{\Gamma_0}$ appears at most once. For these representations the operators Γ_0, $\vec{S}^2$, and S_3 form a canonical set of commuting operators and we can use their eigenvalues as a complete set of quantum numbers for the basis vectors $|\mu j j_3\rangle$, where

$$\Gamma_0|\mu j j_3\rangle = \mu|\mu j j_3\rangle, \qquad \frac{1}{2} S_{ij}S^{ij}|\mu j j_3\rangle = j(j+1)|\mu j j_3\rangle, \qquad S_3|\mu j j_3\rangle = j_3|\mu j j_3\rangle \tag{3.26}$$

By induction, μ, $j(j+1)$, and j_3 are in fact the eigenvalues of the covariant operators $\hat{P}_\nu \Gamma^\nu$, $\frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}$, and Σ_3, and (3.26) holds on the subspace of the rest-frame states. We note that for the Majorana irreducible representations considered in the section above, this new quantum number is redundant, as $\mu = j + \frac{1}{2}$.

For our purposes, we are interested only in the unitary singleton representations for which Γ_0 is bounded from below or from above. This class of representations is characterized by the two numbers

$$\begin{aligned} \mu_{\min}\ (\mu_{\max}) &= \text{lowest (highest) value of } \mu \\ s &= \text{lowest value of } j \end{aligned} \tag{3.27}$$

The eigenvalues of the Casimir operators (3.2), (3.3)

$$\begin{aligned} R &= \text{eigenvalue of } C_2 = \frac{1}{2} S_{AB}S^{AB} \\ P &= \text{eigenvalue of } C_4 = -W_A W^A \end{aligned} \tag{3.28}$$

are given in terms of $\mu_{\min}$ and s. In this class of representations we identify three subclasses:

$$s = 0, \qquad \mu_{\min} \geq \frac{1}{2} \quad \text{with} \quad R = \frac{9}{4} - \left(\mu_{\min} - \frac{3}{2}\right)^2 \tag{3.29a}$$

$$s = \frac{1}{2}, \qquad \mu_{\min} \geq 1 \quad \text{with} \quad R = \frac{3}{2} - \left(\mu_{\min} - \frac{3}{2}\right)^2 \tag{3.29b}$$

$$\begin{aligned} s = 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots \qquad \mu_{\min} = s + 1 \quad \text{with} \quad R &= 2 - 2(\mu_{\min} - 1)^2 \\ &= 2(1 - s^2) \end{aligned} \tag{3.29c}$$

For all three cases, the eigenvalue of the fourth-order Casimir operator is given by

$$P = s(s + 1)[R - (s - 1)(s + 2)]$$

In the discussion of the hadron spectrum (see Section 5), we will use some of these singleton representations and their weight diagrams, which are also called K-types.

4. RELATIVISTIC HAMILTONIAN AND THE DYNAMICS OF INTERIOR OBSERVABLES

In order to model the dynamics of the interior observables, we next have to conjecture the constraint relation between the generators of the symmetry group and SGG [a relativistic generalization of (2.4)]. These relations can be expressed as

$$(P^{\mu}P_{\mu} - M^2(\pi_{\mu}, d_{\mu} \ldots))\phi = 0 \tag{4.1a}$$

and some subsidiary conditions such as

$$L\phi = 0, \qquad \text{where} \quad L = P_{\mu}d^{\mu} \quad \text{and/or} \quad L = P_{\mu}\pi^{\mu} \quad \text{and/or} \quad L = P_{\mu}\Sigma^{\mu\nu} \tag{4.1b}$$

ϕ here is a nonlocal field, and M^2 is a function of the interior observables. As we shall soon see, for different models, one has different functions for M such as $M^2 = \frac{1}{2}\lambda^2\Sigma_{\mu\nu}\Sigma^{\mu\nu}$ for the rotator model and $M^2 = (1/\alpha)\hat{P}_{\mu}\Gamma^{\mu}$ for the oscillator. L eliminates timelike excitations (ghosts), and since our d^{μ}, π^{μ}, and $\Sigma^{\mu\nu}$, defined respectively by (2.5), (2.13), and (2.10), automatically fulfill (4.1b), there will be no ghosts.

Using the theoretical framework of constraint Hamiltonian mechanics (Dirac, 1964), we can obtain the Hamiltonians from the constraint relations (and vice versa).

Many relativistic Hamiltonians have been considered (Aldinger *et al.*, 1983; Bohm *et al.*, 1985a, b). The simplest is that for a structureless mass point:

$$H = v(P_{\mu}P^{\mu} - m_0^2) \tag{4.2}$$

where v is the usual Lagrange multiplier of constraint Hamiltonian mechanics. It can be determined by the choice of the parameter τ for the c.m. proper time with respect to which the equations of motion for the observables are derived.

For the relativistic string, on the other hand, the Hamiltonian is very complicated:

$$H = v\left(P_\mu P^\mu + \frac{1}{\alpha'} \sum_{m=1}^{\infty} \alpha_{-m\mu}\alpha^\mu_{+m} - \text{const}\right) \tag{4.3}$$

The mass operator here is given in terms of infinitely many creation (α^μ_{-m}) and annihilation (α^μ_{+m}) operators, which are not—like our d_μ and π_μ—all spacelike. The difficulties of (4.3), which have their origin mainly in the choice of the relativistic canonical intrinsic positions and momenta, are well known.

4.1. Relativistic Quantum Rotator

For manageable, realistic models, we expect the Hamiltonian to be somewhere in between (4.2) and (4.3). As examples, we discuss a model for a relativistic quantum rotator in this section and a model for a relativistic quantum oscillator in the next.

Let us first note that the names such as relativistic quantum rotator and oscillator are justified mainly by the correspondence between these models and the well-established models such as the elementary relativistic particle, the nonrelativistic quantum (rigid) rotator, and the nonrelativistic harmonic oscillator. In particular, the relativistic quantum rotator (RQR) model, considered as that of a relativistic extended object characterized by an elementary length parameter R, contracts to the relativistic elementary particle model when the length parameter is taken to infinity. In the nonrelativistic contraction limit, i.e., $c \to \infty$, the RQR goes over to the familiar rigid rotator of atomic physics.

The observables for the RQR can be constructed in terms of the interior operators defined in Section 2. At that stage the constraint relations between the generators of the symmetry group and the spectrum-generating group were not imposed, and the interior operators $S_{\mu\nu}$ were assumed to commute with the center-of-mass operators such as P_μ. In accord with this assumption, the position X_μ that appears in (2.7b) can be defined by the following commutation relations:

$$[J_{\mu\nu}, X_\rho] = i(g_{\nu\rho}X_\mu - g_{\mu\rho}X_\nu) \tag{4.4}$$

$$[P_\mu, X_\rho] = ig_{\mu\rho}I \tag{4.5}$$

$$[X_\mu, X_\nu] = 0 \tag{4.6}$$

In analogy to the operator $\hat{P}_\nu$ defined in Section 2, it is convenient to define an operator $\hat{X}_\mu$ by

$$\hat{X}_\mu = X_\mu M \tag{4.7}$$

where, as before, $M = (P_\nu \hat{P}^\nu)^{1/2}$. Then by (4.5), we obtain the following commutation relations:

$$[M, X_\mu] = i\hat{P}_\mu \tag{4.8}$$

In terms of these operators, the total angular momentum operator $J_{\mu\nu}$ in (2.7) can be written as

$$J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu + S_{\mu\nu} = \hat{X}_\mu \hat{P}_\nu - \hat{X}_\nu \hat{P}_\mu + S_{\mu\nu} \tag{4.9}$$

In order to conjecture the Hamiltonian for the RQR model, we now consider the following operator:

$$B_\mu = P_\mu - \lambda \hat{b}_\mu \tag{4.10}$$

where

$$b_\mu = \frac{1}{2}\{J_{\mu\nu}, \hat{P}^\mu\} = \frac{1}{2}\{\hat{X}_\mu \hat{P}_\nu - \hat{X}_\nu \hat{P}_\mu, \hat{P}^\nu\} - \hat{d}_\mu$$

is the center operator of Finkelstein (1949). λ in (4.10) is a constant of dimension MeV, and $\hat{d}_\mu = d_\mu M$. As can be verified in a straightforward manner, $\hat{b}_\mu$ is a Lorentz vector operator with respect to $J_{\mu\nu}$ and fulfills the following commutation relations:

$$[\hat{b}_\mu, \hat{b}_\nu] = iJ_{\mu\nu} \tag{4.11}$$

As a result, B_μ in (4.10), another vector operator, fulfills the commutation relations

$$[B_\mu, B_\nu] = i\lambda^2 J_{\mu\nu} \tag{4.12}$$

which can also be written in a dimensionless form as

$$[\hat{B}_\mu, \hat{B}_\nu] = iJ_{\mu\nu} \tag{4.13}$$

where $\hat{B}_\mu = (1/\lambda)B_\mu \equiv (1/\lambda)P_\mu - \frac{1}{2}\{J_{\mu\nu}, \hat{P}^\nu\}$.

By (4.13), it is clear that the $J_{\mu\nu}$ and the vector operator $\hat{B}_\mu$ generate the group $SO(4, 1)_{\hat{B}_\mu, J_{\mu\nu}}$. Thus the operators $\hat{B}_\mu$ are the generators of the $SO(4, 1)$ rotations, and, in particular, in the dimensional form, the B_μ are the generators of motion along a (4, 1) de Sitter sphere of radius $R = 1/\lambda$. The group $SO(4, 1)_{\hat{B}_\mu, J_{\mu\nu}}$ has a Casimir operator C given by

$$C = \hat{B}_\mu \hat{B}^\mu - \frac{1}{2} J_{\mu\nu} J^{\mu\nu} \tag{4.14}$$

Thereupon, we propound as a basic postulate that, in the same way as the relativistic mass point is characterized by the eigenvalues m^2 of the Casimir operator $P_\mu P^\mu$ of the Poincaré group, the relativistic rotator is the model which is characterized by the eigenvalues $\lambda^2\alpha^2$ of the Casimir operator

$$\lambda^2 C = B_\mu B^\mu - \frac{\lambda^2}{2} J_{\mu\nu} J^{\mu\nu} \tag{4.15}$$

of the group $SO(4, 1)_{B_\mu, J_{\mu\nu}}$.

This leads to the constraint relation

$$B_\mu B^\mu - \frac{\lambda^2}{2} J_{\mu\nu} J^{\mu\nu} - \lambda^2\alpha^2 \approx 0 \tag{4.16}$$

for the RQR, in contrast to the constraint relation

$$P_\mu P^\mu - M^2 \approx 0 \tag{4.17}$$

for the relativistic elementary particle. Notice that the equalities in (4.16) and (4.17) are weak, à la Dirac. This constraint relation, according to the formalism of the constraint Hamiltonian mechanics, leads to the following expression for the Hamiltonian of the RQR:

$$H = v\left[B_\mu B^\mu - \frac{\lambda^2}{2} J_{\mu\nu} J^{\mu\nu} - \lambda^2\alpha^2\right] \approx 0 \tag{4.18}$$

where v is the Lagrange multiplier discussed above.

The expression (4.18) for the Hamiltonian of the RQR can be justified by considering the limiting cases $\lambda \to 0$ and $c \to \infty$. As mentioned above, the parameter λ is the inverse radius of a de Sitter sphere. If the notion that gravitational processes result in a curved space can be extended to strong interactions as well, then we can view these interactions in a certain state of equilibrium as generating a micro de Sitter universe—just as the gravitational processes cause a de Sitter universe. Since the strong interactions are much stronger than the gravitational interactions, we have to consider a much smaller de Sitter universe. The group of motion in such a model, finite in space, infinite in time, is not the Poincaré group, but the 4 + 1 de Sitter group. Therefore, the B_μ, which generate the motion along this curved space, must, in the $\lambda \to 0$ (or $R \to \infty$) limit (i.e., when the curvature of the de Sitter space tends to zero), contract to the observables which generate the motion in the *flat space*, i.e., the P_μ. Hence in the limit $\lambda \to 0$, $SO(4, 1)_{B_\mu, J_\mu}$ must contract to the Poincaré group.

In order to obtain a faithful representation of the Poincaré group, when taking the limit $\lambda \to 0$, we must perform the contractions through a series of representations for which $\alpha \to \infty$ such that

$$\lim_{\substack{\lambda \to 0 \\ \alpha \to \infty}} \lambda^2\alpha^2 = m_0^2 \tag{4.19}$$

In this limit $B_\mu \to P_\mu$, and the Hamiltonian (4.18) reduces to that of relativistic elementary particle, (4.2).

In a similar fashion, in the limit $c \to \infty$, i.e., when the Poincaré group by the Inonu and Wigner (1953) contraction goes into the Galilei group, the Hamiltonian (4.18) reduces to the following expression:

$$H = \frac{\vec{P}^2}{2M} + \frac{\lambda^2}{2M}\vec{\Sigma}^{(\infty)2} + \frac{\lambda^2}{2M}\left(\alpha^2 - \frac{9}{4}\right) \tag{4.20}$$

This is the energy operator of the nonrelativistic rotator (2.4a) up to the arbitrary constant $(\lambda'^2/2M)(\alpha^2 - 9/4)$, if we take

$$I = \frac{1}{\lambda^2}M = R^2M \tag{4.21}$$

and if $\vec{S}$ in (2.4a) represents the angular momentum in the c.m. frame [see (4.48b)].

In order to obtain the mass spectrum, we establish a simpler form of the Hamiltonian (4.18) which explicitly contains the Casimir operators $P_\mu P^\mu$ and $\frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu}$ of the Poincaré group. First consider the Pauli–Lubanski–Bargmann vector $\hat{w}_\mu$ defined by

$$\hat{w}_\mu = \frac{1}{4}S_{\mu\nu\rho\sigma}\hat{P}^\nu J^{\rho\sigma} \tag{4.22}$$

Since $L_{\mu\nu}$ that appears in (2.7a) fulfills the condition $\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\mu L^{\rho\sigma} = 0$, and since $\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu(d^\rho P^\sigma - d^\sigma P^\rho) = 0$, $\hat{w}_\mu$ can also be written as

$$\hat{w}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu S^{\rho\sigma} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\hat{P}^\nu\Sigma^{\rho\sigma} \tag{4.23}$$

and it obeys the supplementary condition $P_\mu\hat{w}^\mu = 0$. Hence,

$$\begin{aligned}\hat{W} = -\hat{w}_\mu\hat{w}^\mu &= \frac{1}{2}S_{\mu\nu}S^{\mu\nu} - \hat{P}^\rho\hat{P}^\sigma S_{\rho\mu}S^\mu_\sigma \\ &= \frac{1}{2}S_{\mu\nu}S^{\mu\nu} - \hat{d}_\mu\hat{d}^\mu\end{aligned} \tag{4.24}$$

or

$$\hat{W} = -\hat{w}_\mu\hat{w}^\mu = \frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} \tag{4.25}$$

By inserting (4.10) into (4.18) and using (4.13) and (4.25), we obtain by a straightforward, albeit lengthy, calculation

$$H = v\left(P_\mu P^\mu + \frac{9}{4}\lambda^2 - \lambda^2\hat{W} - \lambda^2\alpha^2\right) \tag{4.26}$$

Here we see that, if we choose the eigenvalue of the $SO(4, 1)$ Casimir operator to be one of the values of the principal series representation, then the Poincaré group representation, related to it by (4.10), has timelike P_μ, i.e., $P_\mu P^\mu > 0$. This permits the definition of the positive-definite operator M by $M = (P_\mu P^\mu)^{1/2}$.

In order to obtain a mass formula pertaining to the RQR and discuss the dynamics of its observables, we now consider the model in the irreducible Majorana representations of the spectrum-generating group $SO(3, 2)_{S_{\mu\nu},\Gamma_\mu}$ discussed in Section 3. Using the condition (3.12), from (4.23) and (4.25) we obtain, respectively,

$$\hat{w}_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu\hat{d}^\rho\Gamma^\sigma(\hat{P}\cdot\Gamma)^{-1} \tag{4.27}$$

and

$$\hat{W} = \hat{P}^\rho\hat{P}^\sigma\Gamma_\rho\Gamma_\sigma - \frac{1}{4} \tag{4.28}$$

Then from (4.26) we obtain for the Hamiltonian of the RQR in the irreducible Majorana representations the following expression:

$$H = v\left(P_\mu P^\mu - \frac{5}{2}\lambda^2 - \lambda^2(\hat{P}_\rho\Gamma^\rho)^2 - \lambda^2\alpha^2\right) \tag{4.29}$$

This expression, prior to imposing the constraint relation (4.16), can be used to compute the time derivatives of the observables $\mathcal{O}$ defined by

$$\frac{d\mathcal{O}}{d\tau} = \dot{\mathcal{O}} = \frac{1}{i}[\mathcal{O}, H] \tag{4.30}$$

Thus, straightforward calculations give

$$\dot{X}_\mu = 2vP_\mu + v\lambda^2\{\hat{P}\cdot\Gamma, \Gamma_\mu - (\hat{P}\cdot\Gamma)\hat{P}_\mu\}M^{-1} \tag{4.31}$$

and

$$\dot{d}_\mu = v\lambda^2\{\hat{P}\cdot\Gamma, \Gamma_\mu - (\hat{P}\cdot\Gamma)\hat{P}_\mu\}M^{-1} \tag{4.32}$$

To arrive at (4.31), we have used the fact that before the constraint is imposed, X_μ commutes with Γ_σ. From (2.9), we obtain $\dot{Y}_\mu$ in terms of $\dot{X}_\mu$ and $\dot{d}_\mu$:

$$\dot{Y}_\mu = \dot{X}_\mu + \dot{d}_\mu \tag{4.33}$$

Therefore, by using (4.31) and (4.32) in this expression, we find that

$$\dot{Y}_\mu = -2\nu P_\mu \tag{4.34}$$

This result shows that the velocity of the center of mass is parallel and proportional to the momentum p_μ. Although [as a result of the commutation relations (2.10) and (4.6)] the Y_μ do not commute with one another, the operators $\dot{Y}_\mu$ do.

As mentioned in (4.2), we can determine the Lagrange multiplier ν that appears in the Hamiltonian (4.29) if we choose the parameter τ as a way of defining a center-of-mass proper time with respect to which the equations of motion (4.30) are defined. It must be noted, however, that there is no well-defined world line for the center of mass in a quantum mechanical theory.

If we choose in particular the gauge condition

$$\frac{dY_\mu}{d\tau}\frac{dY^\mu}{d\tau} = 1 \tag{4.35}$$

we get from (4.34)

$$\nu = \pm\frac{1}{2}M^{-1} \tag{4.36}$$

where, as before, $M = (P_\mu P^\mu)^{1/2}$.

In accordance with the interpretation of (4.34), the minus sign is the correct choice for ν in (4.36). Hence,

$$\dot{Y}_\mu = P_\mu M^{-1} = \hat{P}_\mu \tag{4.37}$$

Therefore, by (4.36) and (4.32), we find

$$\dot{\hat{d}}_\mu = \dot{d}_\mu M = \frac{\lambda^2}{2M}\{\hat{P}\cdot\Gamma, \Gamma_\mu - (P\cdot\Gamma)\hat{P}_\mu\} \tag{4.38}$$

From this and the obvious results

$$\frac{d}{d\tau}(\hat{P}_\rho\Gamma^\rho) = 0 \qquad \text{and} \qquad \dot{\Gamma}_\mu = \frac{\lambda^2}{2M}\{\hat{P}^\rho\Gamma_\rho, \hat{d}_\mu\}$$

we can calculate the second time derivative of $\hat{d}_\mu$:

$$\ddot{\hat{d}}_\mu = \left(\frac{\lambda^2}{2M}\right)^2 \{\hat{P}^\rho\Gamma_\rho, \{\hat{P}^\sigma\Gamma_\sigma, \hat{d}_\mu\}\} \tag{4.39}$$

For the Majorana irreducible representations, the Wigner basis vectors $|pjj_3\rangle$ are the basis vectors for the space of physical states. Here $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$, for the two half-odd-integer Majorana representations and $j = 0, 1,$

2, . . . , for the two integer representations. Furthermore, from the results of Section 3, we find that in this basis

$$\hat{P}_\mu \Gamma^\mu |pjj_3\rangle = \pm(\text{sign } p_0)\left(j + \frac{1}{2}\right)|pjj_3\rangle \tag{4.40}$$

Therefore, we obtain for the expectation value of (4.39) in this basis the following result:

$$\langle|\ddot{\hat{d}}_\mu|\rangle = -\left(\frac{\lambda^2}{2m}\right)^2 4\left(j + \frac{1}{2}\right)^2 \langle|\hat{d}_\mu|\rangle \tag{4.41}$$

Thus we see that the expectation value of the interior position operator performs rotations with an angular frequency given by

$$\omega = \pm\frac{\lambda^2}{m}\left(j + \frac{1}{2}\right) \tag{4.42}$$

While the expectation value $\langle|Y_\mu|\rangle$ of the center of mass follows a *straight world line* in the direction of that of the center-of-mass momentum $\langle|P_\mu|\rangle$, the expectation value of the position $X_\mu(\langle|X_\mu|\rangle = \langle|Y_\mu|\rangle + \langle|d_\mu|\rangle)$, according to (4.42), demonstrates the *Zitterbewegung*.

These calculations lead to a visualization of the model as an extended object which is rotating about its center of mass whose motion is determined by the Poincaré group generators P_μ. The mass operator $M = (P_\mu P^\mu)^{1/2}$, however, is now a function of the generators of the spectrum-generating group.

When the constraint relation (4.16) is taken between the Wigner basis vectors $|pjj_3\rangle$ in the Majorana irreducible representations, we obtain for the rotator excitations the following mass (squared) spectrum:

$$m^2 = \lambda^2\left(\alpha^2 - \frac{9}{4}\right) + \lambda^2 j(j + 1) \tag{4.43}$$

4.2. Relativistic Quantum Oscillator

Just as the relativistic quantum rotator was a relativistic generalization of the familiar quantum (rigid) rotator of molecular physics, in this section we consider a relativistic generalization of the familiar harmonic oscillator. The observables, again, can be chosen from among the operators defined in Section 2. The theoretical framework and the methodology employed in constructing the relativistic quantum oscillator (RQO) are the same as in the RQR model. Thus we begin by conjecturing the Hamiltonian for the RQO.

The nonrelativistic harmonic oscillator, the conventional string, or the linearly rising Regge trajectories all suggest equal spacing for the mass-squared spectrum. Therefore, according to the rules of constraint Hamiltonian mechanics, we seek to express the mass-squared operator $P_\mu P^\mu$ in terms of an operator whose eigenvalues ν are

$$\nu = 0, 1, 2, \cdots$$

Such an operator appeared in the discussion of the RQR: $\hat{P}_\mu \Gamma^\mu$ in (4.40). Thus we postulate the following operator as the Hamiltonian for the RQR:

$$H = v\left(P_\mu P^\mu - \frac{1}{\alpha'} \hat{P}_\mu \Gamma^\mu\right) \tag{4.44}$$

where, as in (4.18), $v = -1/(2M)$ when the parameter τ is chosen as the proper time in the mean c.m., and where α' is the constant discussed in Section 1.

The expression (4.44) is in fact the simplest choice for the free (without electromagnetic coupling) Hamiltonian with the operators available in our model.

With the operator (4.44) for the Hamiltonian, we can now justify the definition (2.13) of the interior momentum operator by obtaining an operator canonically conjugate to d_μ defined in (2.5). Using the commutation relations (2.11) of the relativistic SGG, we obtain the proper time derivative $\dot{d}_\mu$ of d_μ by calculating the commutator of the interior position with H given in (4.44). The result is

$$\dot{d}_\mu = \frac{1}{i}[d_\mu, H] = -\frac{v}{\alpha' M} \breve{g}^\sigma_\mu \Gamma_\sigma \tag{4.45}$$

Since π_μ should be equal (to within a normalization factor) to $M\dot{d}_\mu$, this justifies the definition (2.13) of the momentum operator as $\pi_\mu = (1/v)\dot{d}_\mu$.

The commutation relations for π_μ can now be calculated in a straightforward way using the commutation relations of the SGG. The results are

$$[\pi_\mu, \pi_\nu] = \frac{-i}{(\alpha' M)^2} \Sigma_{\mu\nu} \qquad \text{and} \qquad [d_\mu, \pi_\nu] = -i\breve{g}_{\mu\nu} \frac{1}{\alpha' M^2} \hat{P}_\rho \Gamma^\rho \tag{4.46}$$

These relations, together with (2.10), are the c.r. of our relativistic oscillator. They replace the conventional relativistic canonical commutation relations (Heisenberg)

$$[d_\mu, d_\nu] = 0, \qquad [\pi_\mu, \pi_\nu] = 0, \qquad \text{and} \qquad [d_\mu, \pi_\nu] = -i\breve{g}_{\mu\nu} \tag{4.47}$$

which lead to the well-known difficulties. The relativistic commutation relations (2.10) and (4.46) of interior position and momentum, which are the c.r. of $SO(3, 2)$ in a disguised form, go into the usual Heisenberg c.r. only in the nonrelativistic limit $c \to \infty$. In this limit the Poincaré group $\mathcal{P}$ goes by Inonu and Wigner (1953) contraction into the Galilei group $\mathcal{G}$, and the $SO(3, 2)$ algebra goes into the algebra of the 3-dimensional nonrelativistic oscillator.[2] This limit can best be seen in the rest frame of the center of mass.

From the definitions of d_μ (2.5), π_μ (2.13), and $\Sigma_{\mu\nu}$ (2.10), we obtain in the rest frame of the c.m., i.e., for $p_\mu = (1, 0, 0, 0)$,

$$d_0 = \pi_0 = 0, \qquad d_m = S_{0_m} M^{-1}, \qquad \pi_m = -\frac{1}{\alpha'} M^{-1}\Gamma_m \tag{4.48a}$$

$$\Sigma_{0m} = 0, \qquad \Sigma_{mn} = S_{mn} \tag{4.48b}$$

where $m, n = 1, 2, 3$. In the nonrelativistic limit (Bohm *et al.*, 1985b) d_m and π_m become commuting operators $d_m^{(\infty)}$ and $\pi_m^{(\infty)}$ fulfilling the Heisenberg c.r. (2.3a). The constraint relation (4.44) contracts in this limit to the following constraint:

$$H^{\rm osc} = \frac{1}{2m}\vec{P}^2 + \frac{1}{4m}\vec{\pi}^{(\infty)2} + \frac{1}{\alpha'^2}\vec{d}^{(\infty)2} \tag{4.49}$$

This limit for the particular representation $D(1, 0)$ of $SO(3, 2)$ provides the inspiration for some of the above choices and justifies the name relativistic oscillator for the model given by (4.44).

As we saw in Section 3, there are many other irreducible representations of $SO(3, 2)$. In order to determine the spectrum of the relativistic extended object, we have to first choose the right class of representations, and then, inside this class, the particular irreducible representation which describes our system. In order to allow for the oscillations, we have to go to a larger representation space of $SO(3, 2)_{S_{\mu\nu},\Gamma_\mu}$ than the irreducible Majorana representation spaces considered for the relativistic rotator. We will choose the class of representations which are conventionally denoted by $D(s + 1, s)$, where s can take the values $s = 0$, or $\frac{1}{2}$ or 1 or $\frac{3}{2} \ldots$, for the following reasons:

1. The representation $D(1, 0)$ goes in the limit $c \to \infty$ into the representation of the ordinary, three-dimensional spinless nonrelativistic oscillator.

2. The number s that characterizes each of these representations is identical to the hadronic total quark spin number.

Thus for the ρ-meson and the other meson resonances with higher values of hadron spin j which belong to the same Regge trajectory, we should choose the representation $D(2, 1)$ because all these resonances have total quark spin $s = 1$. For the nucleon and the nucleon resonances, which have total quark spin $s = \frac{1}{2}$, we should choose $D(\frac{3}{2}, \frac{1}{2})$.

These results suggest the following picture for our relativistic model for the extended object. The object consists of a quark–antiquark (or quark–

[2] This was known before this model was developed. See, e.g. Celeghin and Tarlini, (1982).

diquark) pair connected by a vibrating and rotating flux tube. The interior position operators d_μ can be considered as defining the "length and direction" of the flux tube. The collective motion then consists of the vibrations of the tube length and the rotations (about the c.m.) of the tube axis. This picture is more accurate in the classical and nonrelativistic limits, where the notions of trajectory and rigidity, respectively, acquire their usual meanings. The spectra of mass and spin levels of hadrons are then obtained as these vibrational and rotational excitations of this relativistic flux tube.

5. HADRON SPECTRUM

Since a hadron in this model is a vibrationally (and/or rotationally) excited state of the relativistic extended flux tube, we have to determine the spectrum of the vibrational quantum number μ (and the rotational quantum number j) in an irreducible representation of the relativistic spectrum-generating group. From the discussion in Section 3, for the representation $D(1, 0)$ one can see that

$$\mu = \text{eigenvalue of } (\hat{P}_\mu \Gamma^\mu) \overset{\mathrm{R}}{=} \text{eigenvalue of } \Gamma^0 \tag{5.1}$$

is the vibrational quantum number, and j with

$$j(j + 1) = \text{eigenvalue of } (\hat{W}) \overset{\mathrm{R}}{=} \text{eigenvalue of } \vec{S}^2 \tag{5.2}$$

is the rotational quantum number. The R above the equality sign in the above equations indicates that these equalities hold in the rest frame.

Indeed the diagram in Fig. 1 is the energy diagram of the 3-dimensional harmonic oscillator, but in terms of the mathematics of $SO(3, 2)$, it is identical to the weight diagram (K-type) of the irreducible representation $D(1, 0)$.

Mathematically, the numbers ($\mu_0 = s + 1$, $j_0 = s$) that characterize the irreducible representations $D(s + 1, s)$ are the lowest weights. The weights are the ordered pairs (μ, j) which characterize the irreducible representation of the maximally compact subgroup $K = SO(2)_{\Gamma^0} \times SO(3)_{S^{ij}}$ of $SO(3, 2)$ which can occur in the representation $D(\mu_0, j_0)$. In terms of the operators, the weights are given by (5.1) and (5.2).

For physical reasons ("stability of matter"), we are interested only in the irreps of the SGG whose weights are semibounded. The $D(s + 1, s)$ are a particular class of these representations with semibounded weights.

The weights of the K-type $D(2, 1)$, are shown by the dots in Fig. 2 (Bohm, 1992; Bohm and Wickramasekara, 1997). The states with $j = \mu = 1 \equiv \nu$ (the vibrational quantum number), which would in the terminology of nuclear physics be the yrast states, represent the hadrons on the Regge trajectory, while the ones with $j < \nu$ represent the "daughters."

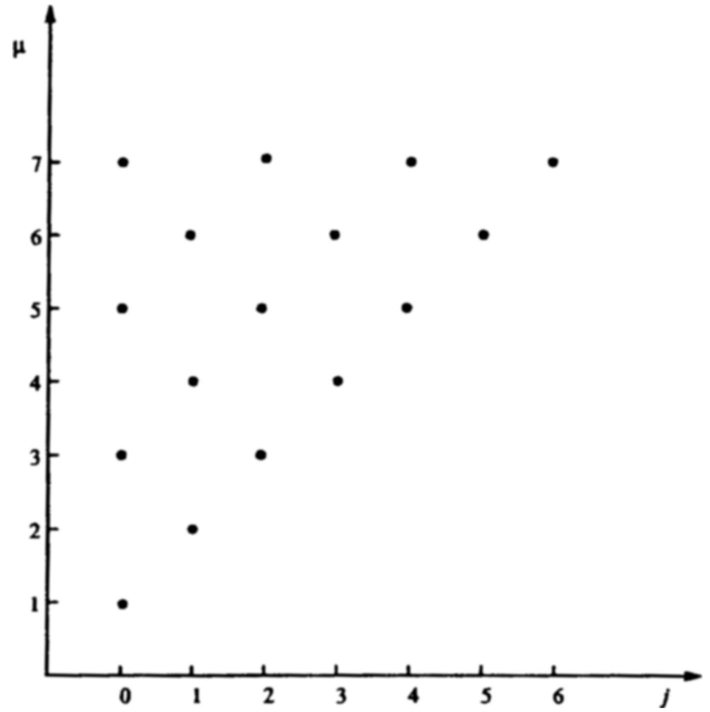

Fig. 1. The weight diagram of the representation $D(1, 0)$, identical to the energy diagram of the three-dimensional nonrelativistic oscillator.

The weight diagram (K-type) of representation $D(s + 1, s)$ gives a graphical representation of the spectrum (μ, j) in that D. In the K-type of $D(1, 0)$ shown in Fig. 1, each dot gives the value of a pair (μ, j). So to each dot corresponds a hadron state with the vibrational quantum number μ and the rotational quantum number j. Empirically, $D(1, 0)$ is not the correct representation for the Regge trajectory of the ρ-meson, because the ρ-trajectory does not have a $j = 0$ ground state. But since s is understood as the total quark spin, which for the ρ-meson is equal to 1, not $D(1, 0)$, but $D(2, 1)$ must be the right representation for the ρ-trajectory. Similarly, $D(\frac{3}{2}, \frac{1}{2})$ with $s = \frac{1}{2}$ must be the correct representation for the nucleon trajectory.

Thus the spectrum of the vibrational and rotational quantum numbers of the "relativistically" vibrating and rotating flux tube is determined by the weight diagrams of the relativistic SGG [K-type of $SO(3, 2)$]. The mass spectrum is obtained from the relativistic Hamiltonian, i.e., from the constraint relation as in the formalism of constraint Hamiltonian mechanics.

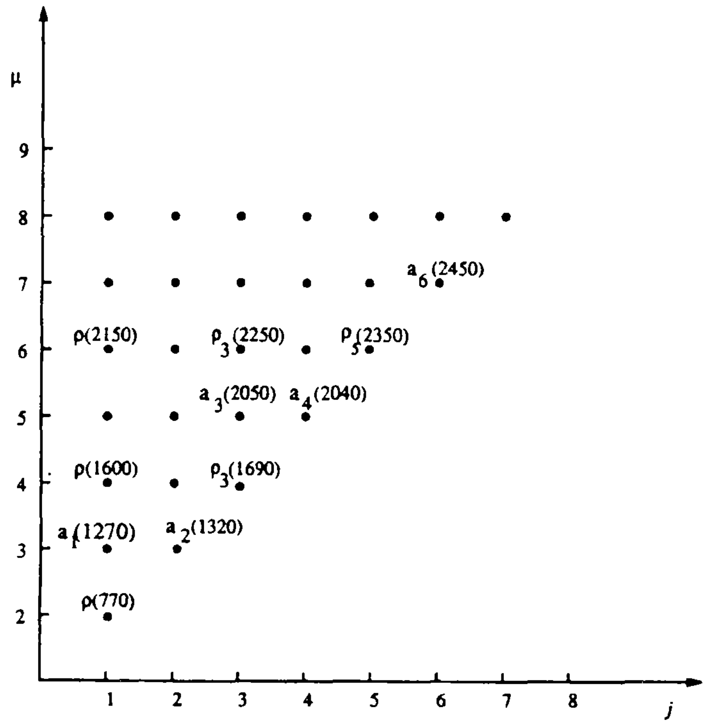

Fig. 2. The weight diagram of the representation $D(2, 1)$ with the assignment of mesons. Each dot represents a meson with spin j and vibrational quantum number μ. The mesons assigned are from the Particle Data Table (1990).

The Hamiltonian (4.44) of the relativistic vibrator leads to the master equation

$$\left(\frac{1}{\alpha'} \hat{P}_\mu \Gamma^\mu - P_\mu P^\mu\right)|\psi\rangle = 0 \tag{5.3}$$

which is reminiscent of the Dirac equation

$$\left(\frac{1}{\alpha} \frac{1}{m_0} P_\mu \gamma^\mu - \frac{1}{\alpha}\right)|\psi\rangle = 0$$

with the $\frac{1}{2}\gamma^\mu$ replaced by Γ^μ. Note that, in particular, $\frac{1}{2}\gamma^0 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ is replaced by Γ^0, which is an infinite-dimensional operator with a nontrivial spectrum (Mukunda *et al.*, 1982). The spectrum of the mass-squared operator $P_\mu P^\mu$ is then uniquely and completely determined from the spectrum of $\hat{P}_\mu \Gamma^\mu$.

Just as for the RQR model, calling m^2 the eigenvalues of the operator $P_\mu P^\mu = M^2$, one obtains from (5.3) the mass formula

$$m^2 = m_0^2 + \frac{1}{\alpha'} \nu, \qquad \nu = \mu - 1 = 1, 2, 3, \dots, \qquad j = 1, 2, \dots, \nu \tag{5.4}$$

This means that all mass levels with the same vibrational quantum number have the same mass.

For the Regge trajectory $j = \nu$ (yrast states), one obtains

$$m^2 = m_0^2 + \frac{1}{\alpha'} j \tag{5.5}$$

In order to lift the mass degeneracy in j one has to use the Hamiltonian of the relativistic vibrating rotator. In the same way as the oscillator and rotator models are combined in molecular physics, we propose to combine the RQR and RQO Hamiltonians to obtain a Hamiltonian for the "relativistically" rotating and oscillating extended quantum system. Hence,

$$H = \nu\left(P_\mu P^\mu - \frac{1}{\alpha'} \hat{P}_{\mu;}\Gamma^\mu - \lambda^2 \hat{W} - m_0^2\right) \tag{5.6}$$

This Hamiltonian describes the rotations as well as the oscillations of the flux tube. From the spectra of the operators (5.1) and (5.2) one then obtains for the relativistic Hamiltonian (5.6) the mass spectrum

$$\begin{aligned} m^2 &= m_0^2 + \frac{1}{\alpha'} \nu + \lambda^2 j(j + 1), \\ \nu &= \mu - 1 = 1, 2, 3, \dots, \\ j &= 1, 2, \dots, \nu \end{aligned} \tag{5.7}$$

The numbers attached to the dots in Fig. 2, for example, are the experimental masses of the meson resonances associated to that state with quantum numbers (ν, j) of the dot.

These masses in Fig. 2 are fitted to (5.7) and from this fit, we find $1/\alpha = 1.04\ (\mathrm{GeV})^2$ and $\lambda^2 = 0.02\ (\mathrm{GeV})^2$. Equally good fits have been obtained with this model both for, as evident from Fig. 3, the nucleon resonances and for the hadron towers of different flavors (with α' and λ^2 depending on the flavor).

Therewith we see that the vibrational and rotational excitations of the interior degrees of freedom of an extended relativistic object are described as different states of an irreducible representation of the spectrum-generating group. These states are represented by irreducible representation spaces

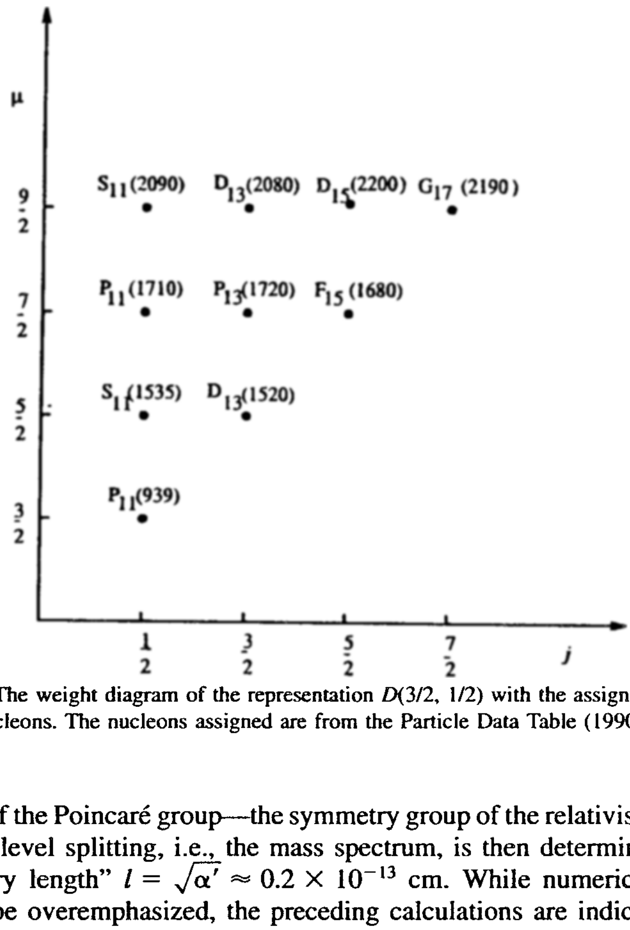

Fig. 3. The weight diagram of the representation $D(3/2, 1/2)$ with the assignment of nucleons. The nucleons assigned are from the Particle Data Table (1990).

$\mathcal{H}^{\nu}(m, j)$ of the Poincaré group—the symmetry group of the relativistic space-time. The level splitting, i.e., the mass spectrum, is then determined by an "elementary length" $l = \sqrt{\alpha'} \approx 0.2 \times 10^{-13}$ cm. While numerical results must not be overemphasized, the preceding calculations are indicative that the spectra of hadrons can be obtained in much the same way as in molecular and nuclear physics, but with a relativistic theory. The one presented here arises from a union of Heisenberg's idea of an *elementary length* with the idea that the mass spectrum of hadrons has a group-theoretic interpretation.

ACKNOWLEDGMENTS

The author wishes to acknowledge his intellectual debt to Arno Bohm. Many thanks are also due to George Pronko for his insightful comments on this work, especially from a canonical approach. Mario Castagnino, Edgard Gunzig, and Mady Smets deserve special thanks for the valuable opportunity provided at the beautiful Peyresq Conference to present this work.

REFERENCES

Aldinger, R. R., *et al.*, (1983). *Physical Review D*, **28**, 3020.
Aldinger, R. R., *et al.*, (1984). *Physical Review D*, **29**, 2828.
Barut, A. O., and Bohm, A. (1965). *Physical Review*, **139B**, 1107.
Bohm, A. (1967). In *Lectures in Theoretical Physics*, E. Britten and A. O. Barut, eds., Gordon and Breach, New York.
Bohm, A. (1973). In *Studies in Mathematical Physics*, A. O. Barut, ed., Reidel, Boston.
Bohm, A. (1979). *Quantum Mechanics: Foundations and Applications*, 2nd ed., Springer, Berlin.
Bohm, A. (1992). In *Proceedings of the International Symposium on Extended Objects and Bound Systems*, Karuizawa, Japan.
Bohm, A., and Wickramasekara, S. (1997). *Turkish Journal of Physics*, **21**, 289.
Bohm, A., *et al.*, (1985a). *Physical Review D*, **31**, 2304.
Bohm, A., *et al.* (1985a). *Physical Review D*, **32**, 791.
Bohm, A., Ne'eman, Y., and Barut, A. O. (1988). *Dynamical Groups and Spectrum Generating Algebras*, World Scientific, Singapore.
Bohm, A., Kendrick, B., Loewe, M. E., and Boya, L. J. (1992). *Journal of Mathematical Physics*, **33**, 977.
Celeghin, E., and Tarlini, M. (1982), *Nuevo Cimento*, **68B**, 133.
Corben, H. C. (1968). *Classical and Quantum Theories of Spinning Particles*, Holden-Day, San Francisco.
Dasher, R. F., and Gell-Mann, M. (1965). *Physics Letters*, **17**, 142.
Dothan, Y., Gell-Mann, M., and Ne'eman, Y. (1965). *Physics Letters*, **17**, 148.
Dirac, P. A. M. (1964). *Lectures on Quantum Mechanics*, Yeshiva University Press, New York.
Finkelstein, R. F. (1949). *Physical Review*, **75**, 1079.
Heisenberg, W. (1936). *Zeitschrift für Physik*, **101**, 533.
Heisenberg, W. (1938). *Zeitschrift für Physik*, **110**, 251.
Iachello, F., Levine, R. D., and van Rodsmalen, O. S. (1982). *Journal of Chemical Physics*, **77**, 3047.
Inonu, E., and Wigner, E. P. (1953). *Proceedings of the National Academy of Science USA*, **39**, 510.
Mathison, M. (1937). *Acta Physica Polonica*, **6**, 163.
Mukunda, N., van Dam, H., and Biedenharn, L. C. (1982). *Relativistic Models of Extended Hadrons Obeying a Mass-Spin Trajectory Constraint*, Springer, Berlin.
Particle Data Table (1990). *Physics Letters B*, **239**.
Takabayasi, T. (1960). *Progress of Theoretical Physics*, **23**, 915.
Takabayasi, T., and Takeuchi, K. (1978). *Progress of Theoretical Physics*, **59**, 2133.
Wigner, E. P. (1939). *Annals of Mathematics*, **40**, 149.